Corrigé de la question 1 de l'exercice 10 – Fiche « Ensembles et nombres »

1.

$$\frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}} \geq \frac{2}{\sqrt{a}+\sqrt{b}} \Leftrightarrow \frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}} - \frac{2}{\sqrt{a}+\sqrt{b}} \geq 0$$

$$\Leftrightarrow \frac{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}{\sqrt{2}(\sqrt{a}+\sqrt{b})} - \frac{2\sqrt{2}}{\sqrt{2}(\sqrt{a}+\sqrt{b})} \geq 0$$

$$\Leftrightarrow \frac{a-b-2\sqrt{2}}{\underbrace{\sqrt{2}(\sqrt{a}+\sqrt{b})}_{>0}} \geq 0 \Leftrightarrow a-b-2\sqrt{2} \geq 0 \Leftrightarrow \boldsymbol{a \geq} b+2\underbrace{\sqrt{2}}_{>1} \boldsymbol{\geq b+2}$$

$$\frac{a+b}{4} - \frac{ab}{a+b} = \frac{(a+b)(a+b)-4ab}{4(a+b)} = \frac{a^2+2ab+b^2-4ab}{4(a+b)} = \frac{a^2-2ab+b^2}{4(a+b)}$$

$$= \frac{(a-b)^2}{4(a+b)} \geq 0 \Leftrightarrow \frac{a+b}{4} \geq \frac{ab}{a+b}$$

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} \leq \frac{a+b}{4} + \frac{b+c}{4} + \frac{c+a}{4}$$

$$\leq \frac{a+b+b+c+c+a}{4}$$

$$\leq \frac{2a+2b+2c}{4}$$

$$\boldsymbol{\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} \leq \frac{a+b+c}{2}}$$

2. D'une part, $\boldsymbol{0 \leq \frac{a+b}{1+ab}}$ car $a, b \geq 0$. D'autre part :

$$1 - \frac{a+b}{1+ab} = \frac{1+ab-a-b}{1+ab} = \frac{1-b-a(1-b)}{1+ab} = \frac{(1-b)(1-a)}{1+ab} \geq 0$$

car $1+ab > 0,\ b \leq 1 \Leftrightarrow 1-b \geq 0$ et $a \leq 1 \Leftrightarrow 1-a \geq 0$

Soit : $\boldsymbol{1 \geq \frac{a+b}{1+ab}}$.

$$\boldsymbol{\frac{a+b}{1+ab} \in [0\,;1]}$$